Journal of Statistical Physics, Vol. 80, Nos. 3/4, 1995

Ballistic Annihilation and Deterministic Surface Growth

Vladimir Belitsky[1] **and Pablo A. Ferrari**[1]

Received October 10, 1994; final February 10, 1995

A model of deterministic surface growth studied by Krug and Spohn, a model of the annihilating reaction $A+B \to inert$ studied by Elskens and Frisch, a one-dimensional three-color cyclic cellular automaton studied by Fisch, and a particular automaton that has the number 184 in the classification of Wolfram can be studied via a cellular automaton with stochastic initial data called ballistic annihilation. This automaton is defined by the following rules: At time $t=0$, one particle is put at each integer point of $\mathbb{R}$. To each particle, a velocity is assigned in such a way that it may be either $+1$ or -1 with probabilities 1/2, independent of the velocities of the other particles. As time goes on, each particle moves along $\mathbb{R}$ at the velocity assigned to it and annihilates when it collides with another particle. In the present paper we compute the distribution of this automaton for each time $t \in \mathbb{N}$. We then use this result to obtain the hydrodynamic limit for the surface profile from the model of deterministic surface growth mentioned above. We also show the relation of this limit process to the process which we call moving local minimum of Brownian motion. The latter is the process $B_x^{\min}$, $x \in \mathbb{R}$, defined by $B_x^{\min} := \min\{B_y;\ x-1 \leqslant y \leqslant x+1\}$ for every $x \in \mathbb{R}$, where B_x, $x \in \mathbb{R}$, is the standard Brownian motion with $B_0 = 0$.

KEY WORDS: Cellular automaton; deterministic model of surface growth; ballistic annihilation; three-color cyclic cellular automaton; annihilating two-species reaction; hydrodynmic limit; moving local minimum of Brownian motion.

1. INTRODUCTION

For $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $X_i(n) \in \{-1, 0, +1\}$ express the presence of a particle at site i and the velocity of this particle at time n; namely, $X_i(n) = 0$

[1] Instituto de Matematica e Estatística, Universidade de São Paulo, Cx. Postal 66281 CEP 05389-970, São Paulo, SP, Brazil. E-mail: vbel1@ime.usp.br, pablo@ime.usp.br

0022-4715/95/0800-0517$07.50/0

means that at time n there is no particle at the site i; $|X_i(n)| = 1$ means that at time n a particle occupies this site and then $X_i(n)$ is its velocity. Denote $X(n) := \{X_i(n),\ i \in \mathbb{Z}\}$. Let us introduce the following rule, which relates $X(n+1)$ deterministically to $X(n)$:

(i) from time n to time $n+1$ each particle which was present in $X(n)$ moves in the direction of its velocity along $\mathbb{R}$

(ii) if, while moving, a particle collides with another particle, then both are annihilated and disappear (1.1)

Let $(\Omega, \mathscr{F})$ denote an abstract probability space on which $X_i(0)$, $i \in \mathbb{Z}$, take their values such that

$$X_i(0), \quad i \in \mathbb{Z}, \text{ are i.i.d. random variables with } \mathbb{P}[X_i(0) = \pm 1] = 1/2 \tag{1.2}$$

The initial distribution (1.2) and the rules (i)–(ii) of (1.1) imposed for every $n \in \mathbb{N}$ define on Ω the process $\{X(n),\ n \in \mathbb{N}\}$ which we call *ballistic annihilation* (BA). BA is a discrete-time cellular automaton with stochastic initial data.

We will now use the process $X(n)$, $n \in \mathbb{N}$, to construct another discrete-time process $S_{\cdot}(n)$, $n \in \mathbb{N}$. This is done in the following manner:

for each $\omega \in \Omega$, $S_{\cdot}(n)[\omega]$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $S_0(n)[\omega] = 0$ and if $X(n)[\omega]$ contains a particle with a positive (resp., negative) velocity at a site $i \in \mathbb{Z}$, then $S_{\cdot}(n)[\omega]$ increases (resp., decreases) linearly with the tangent 1 between the abscissas i and $i+1$; if there is no particle at i in $X(n)[\omega]$, then $S_{\cdot}(n)[\omega]$ is constant between the abscissas i and $i+1$ (1.3)

The process just defined will be called an *integrated process*. Definitely, $S_{\cdot}(n)$ reflects completely the distribution of particles and their velocities in the BA at time n. However, the description of the limit $(n \to \infty)$ law of $S_{\cdot}(n)$ is much easier than that of the limit law of $X(n)$ and thus is preferred in this paper. The relation of $S_{\cdot}(n)[\omega]$ to $S_{\cdot}(0)[\omega]$ is the following: for each $\omega \in \Omega$ and each $n \geqslant 1$, the function $\bar{S}_{\cdot}(n)[\omega]$ defined by

$$\bar{S}_x(n)[\omega] := \min\{S_y(n-1)[\omega],\ x-1 \leqslant y \leqslant x+1\}, \qquad x \in \mathbb{R} \tag{1.4}$$

has the same shape as $S_{\cdot}(n)[\omega]$ and one may be obtained from the other by a vertical shift. The latter assertion is checked straightforwardly.

Iterating, then, one obtains that $S_{.}(n)[\omega]$ has the same shape as the function whose value on $\omega \in \Omega$ and $x \in \mathbb{R}$ is defined by

$$\min\{S_y(0)[\omega], x-n \leqslant y \leqslant x+n\} \tag{1.5}$$

and one may be obtained from the other by a vertical shift.

An application of this fact will be presented in a moment.

The contents of the present paper are the following:

Theorem 1 provides the distribution of the particles in the BA at each instant $n \in \mathbb{N}$. It states that this distribution is invariant with respect to translations of $\mathbb{Z}$ and that given a particle at a site $i \in \mathbb{Z}$, the distributions of the particles to its left and to its right are independent. To complete the picture it thus suffices to present the law of the interparticle distance. Theorem 1 states that if there is a particle at time n at a given site, the probability the particle to its right has the opposite velocity is proportional to $(\sqrt{n})^{-1}$. Consequently, one will see $\mathbb{Z}$ divided into portions in such a way that each one contains solely particles of the same velocity, a cluster. The typical number of particles in a cluster is of order $\sqrt{n}$. Within a cluster the interparticle distance is distributed as the random variable $\sigma(n)$ defined in (2.1) below; a cluster of particles with negative velocity is separated from the cluster to its right with the particles of positive velocity by a particle-free portion of $\mathbb{Z}$ of the random length $2n+\sigma(n)$, while in the opposite case, this length is distributed as $\theta(n)+\sigma(n)$, where the random variable $\theta(n)$ is independent of $\sigma(n)$ and is specified in (2.4) below. The distributions of $\sigma(n)$ and $\theta(n)$ are calculated explicitly through the times of return to zero of a simple symmetric one-dimensional random walk. As we will show in the proof of Theorem 2, $\mathbb{E}\sigma(n)$ is proportional to $\sqrt{n}$, so the mean spatial extension of a cluster is of order n.

Part (a) of Theorem 2 presents the limit (time $\to \infty$) distribution of particles in BA in terms of the limit law of the corresponding integrated process. We consider $S_{.}(n)$ conditioned to the event that at time n there is a particle with positive velocity at the site 0 and the nearest particle to its left has negative velocity. We show that this conditional distribution rescaled by $(2n)^{-1}$ along the abscissa and by $n^{-1/2}$ along the ordinate converges (as $n \to \infty$) to the process $\Psi_{.}$ which we explicitly construct in Section 2.2. A generic trajectory of this process is presented in Fig. 1. The increasing portion of a trajectory of Ψ corresponds to a cluster of particles with positive velocity, while its decreasing portion corresponds to the opposite velocity. The "valleys" of Ψ have length 1 because $\sigma(n)/(2n)$ is infinitesimal; however, the "plateaus" are random because $\theta(n)/(2n)$ has a nontrivial limit $\hat{\theta}$ from (2.13) [this limit has been computed in Eq. (3.25) of ref. 8 through approximation of $\theta(n)$].

Part (b) of Theorem 2 relates $\Psi_{.}$ to the process $B_{.}^{\min,t}$ that is constructed by transforming the standard one-dimensional Brownian motion B_x, $x \in \mathbb{R}$ ($B_0 = 0$), in the following way:

$$B_x^{\min,t} := \min\{B_y : x - t/2 \leqslant y \leqslant x + t/2\}, \qquad t \in \mathbb{R} \tag{1.6}$$

We call the process defined in (1.6) the *moving local minimum of the Brownian motion.* Its relation to the process $\Psi_{.}$ stems from two facts: First, as we have shown in (1.4), (1.5), the shape of $S_{.}(n)$ may be obtained from $S_{.}(0)$ using the transformation that is of the same nature as the one employed to construct $B_{.}^{\min,t}$ from $B_{.}$. Second, the rescale which brings $S_{.}(n)$ to $\Psi_{.}$, being applied to $S_{.}(0)$, brings it to the Brownian motion.

Our interest in the BA was motivated in part by the work of Krug and Spohn.[8] There, the process $\{S_{.}(n), n \in \mathbb{N}\}$ was considered as a model of surface growth. The authors observed that the study of their process is equivalent to the study of the process $\{X(n), n \in \mathbb{N}\}$ that in turn had appeared in the mathematical literature as a model of the irreversible reaction $A + B \to$ *inert*. This model is usually called *annihilating kinetics*. Some results and an extensive list of references on this issue may be found in the work of Elskens and Frisch.[3] Krug and Spohn also noticed that the dynamics of their process was equivalent to that of a particlar cellular automaton that has the number 184 in the classification of Wolfram.[11] Then Fisch[5] studied the same process to obtain results about another process with equivalent dynamics, the so-called one-dimensional three-color cyclic cellular automaton.

Krug and Spohn computed the decay of density of particles in the BA and the two- and three-point correlation functions. They also discussed the shape of $S_{.}(n)$ and the transformation it undergoes when $n \to \infty$ (in this respect, see Remark 1 in Section 2.2 below). The results obtained in refs. 3 and 5 provide the rate of the asymptotic decay of the density. Their results were also partially extended to the case when the initial distribution of particles forms a renewal process and the particle velocities are chosen independently with equal probabilities. The case of a general initial distribution of particles was studied by Ben-Naim *et al.*[3]; however, the argument in this work is based on the mean-field approximation, which is different from that employed in the references we mentioned above.

We must emphasize that our proof of Theorem 1 as well as those from the papers which investigate the dynamics (1.1) are based essentially on the following property of the BA: the time elapsed till a given particle annihilates may be expressed through the time of the first return to the origin of a random walk in $\mathbb{Z}$ (this will be stated exactly in Assertion 1 of Section 3). This random walk is simple and symmetric for the initial distribution (1.2).

2. RESULTS

2.1. The Distribution of the Process for each $n \in \mathbb{N}$

This is given by Theorem 1 of this section. It states that the point process induced by $\{i \in \mathbb{Z} : X_i(n) = 1\}$ is a space-homogeneous renewal process; it then gives the distribution of the interrenewal times and the distribution of the particles with negative velocity between successive renewals. The section terminates with a justification of why Theorem 1 indeed gives a complete description of the BA at time n.

Throughout this paper, $u_l(f_l)$ stands for the probability that a simple discrete-time random walk on $\mathbb{Z}$ starting from zero returns to the origin (for the first time, respectively) at the epoch l.

For each $n \in \mathbb{N}$, we introduce a random variable $\sigma(n)$ by [in (2.1) below and throughout the paper, a sum is assumed to be zero when its upper limit of summation is less than the lower one]

$$\sigma(n) := 1 + \sum_{i=1}^{\lambda(n)} g_i(n) \tag{2.1}$$

where $g_i(n)$, $i = 1, 2,...$, is a sequence of i.i.d. random variables with

$$\mathbb{P}[g_1(n) = l] = f_l(1 - u_{2n})^{-1}, \qquad l = 2, 4,..., 2n \tag{2.2}$$

and the random variable $\lambda(n)$ is independent of $g_i(n)$, $i = 1, 2,...$, and is specified through

$$\mathbb{P}[\lambda(n) = m] = \frac{1 + u_{2n}}{2}\left(\frac{1 - u_{2n}}{2}\right)^m, \qquad m = 0, 1,... \tag{2.3}$$

We also introduce random variables $\theta(n)$, $n \in \mathbb{N}$, by

$$\mathbb{P}[\theta(n) = l - 1] = (u_{2n}^2)^{-1}\,(f_l f_{2n} + f_{l+2} f_{2n-2} + \cdots + f_{2n} f_l), \qquad l = 2,..., 2n \tag{2.4}$$

Theorem 1. Assume (1.2) is the initial distribution of particles in BA. Let n be an arbitrary positive integer. Then:

(i) The distribution of $X(n)$ is invariant with respect to translations of $\mathbb{Z}$.

(ii) Assume that for some $i \in \mathbb{Z}$, the event $C = \{X_i(n) = 1\}$ holds. Let A and B be two events which depend on $X_j(n)$, $j < i$, and $X_j(n)$, $j > i$, respectively, and such that $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Then

$$\mathbb{P}[A \mid C \cap B] = \mathbb{P}[A \mid C] \quad \text{and} \quad \mathbb{P}[B \mid C \cap A] = \mathbb{P}[B \mid C] \tag{2.5}$$

The relations (2.5) are also true for $C = \{X_i(n) = -1\}$.

(iii) For $i \in \mathbb{Z}$, denote by $R(i)$ the position of the first particle to the right of the site i at time n. Then, independently of i,

$$p_n := \mathbb{P}[X_i(n) = X_{R(i)}(n) \mid |X_i(n)| = 1]$$
$$= 1 - \mathbb{P}[X_i(n) = -X_{R(i)}(n) \mid |X_i(n)| = 1] = \frac{1}{1+u_{2n}} \tag{2.6}$$

(iv) Under the notation of (iii) above, for each $k \in \mathbb{N}$, independently of i,

$$\begin{aligned} \mathbb{P}[R(i)-i=k \mid X_i(n) = X_{R(i)}(n) = -1] &= \mathbb{P}[\sigma(n)=k] \\ \mathbb{P}[R(i)-i=k \mid X_i(n) = X_{R(i)}(n) = 1] &= \mathbb{P}[\sigma(n)=k] \\ \mathbb{P}[R(i)-i=k \mid X_i(n) = -X_{R(i)}(n) = -1] &= \mathbb{P}[2n+\sigma(n)=k] \\ \mathbb{P}[R(i)-i=k \mid X_i(n) = -X_{R(i)}(n) = 1] &= \mathbb{P}[\theta(n)+\sigma(n)=k] \end{aligned} \tag{2.7}$$

The above theorem describes completely the distribution of particles and their velocities in the BA at each time $n \in \mathbb{N}$. This fact is substantiated by the following reasoning.

Let $L(i)$ denote the position of the first particle to the left of the site i. For each n, the set $\mathscr{I}(n) := \{i : X_i(n) = 1 \text{ and } X_{L(i)}(n) = -1\}$ is almost surely not empty (this will follow from the argument we use to prove Theorem 1) and the points of this set form a stationary point process on $\mathbb{Z}$, due to the above theorem. The same theorem gives the distribution of the BA at time n conditioned to the event $0 \in \mathscr{I}(n)$. By standard Palm distribution theory, this determines uniquely the distribution of the BA provided the mean distance between the points of the set $\mathscr{I}(n)$ is finite (for each fixed n). The latter is true: indeed, due to (ii)–(iii) of the above theorem, $p_n^k(1-p_n)$ is the probability that given a particle at time n, one will find exactly k ($k = 0, 1, 2,...$) particles with the same velocity to its right before a first particle with the oppsite velocity is met. Using then (iv) and the definitions (2.1)–(2.4), one finds that the expectation of interest equals

$$2\left(\sum_{k=0}^{\infty} k p_n^k (1-p_n)\right) \mathbb{E}\sigma(n) + \mathbb{E}[2n + 2\sigma(n)] + \mathbb{E}[\theta(n) + 2\sigma(n)]$$

and is finite.

2.2. The Limit Law of the Process

This section discusses the limit behavior of the BA. We give an explicit construction of the law to which the integrated process that corresponds to the BA converges as $n \to \infty$ when being appropriately rescaled. We then use a representation of the dynamics of the BA by a simple transformation of a functional space to show that this law relates to the law of $B_{\cdot}^{\min,t}$ defined in (1.6).

We define the family of operators $\{M_r\}_{r \geqslant 0}$ which act in the following manner:

$$(M_r f_\cdot)_x := \min\{f_x : x - r \leqslant y \leqslant x + r\}, \qquad \forall x \in \mathbb{R}, \quad \forall f_\cdot : \mathbb{R} \to \mathbb{R} \text{ continuous} \tag{2.8}$$

For each $n \in \mathbb{N}$, we then define a random function $\hat{S}_\cdot(n)$: $\mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} &\hat{S}_x(0)[\omega] := S_x(0)[\omega], \qquad x \in \mathbb{R}, \quad \omega \in \Omega, \\ &\qquad\qquad \text{where } S_\cdot(0) \text{ is from (1.3); and for } n \geqslant 1, \\ &\hat{S}_x(n)[\omega] := (M_1 \hat{S}_\cdot(n-1)[\omega])_x, \qquad x \in \mathbb{R}, \quad \omega \in \Omega \end{aligned} \tag{2.9}$$

An important observation is that for each n, ω, and $x \in \mathbb{N}$,

$$\hat{S}_{x+1}(n)[\omega] - \hat{S}_x(n)[\omega] = X_x(n)[\omega] \tag{2.10}$$

The above relation holds for $n = 0$ by the definitions (1.3) and (2.9). One then verifies (2.10) inductively in n by comparing the way $X(n)[\omega]$ is obtained from $X(n-1)[\omega]$ with the way $\hat{S}_\cdot(n)[\omega]$ is obtained from $\hat{S}_\cdot(n-1)[\omega]$. It is also straightforward to see that $\hat{S}_\cdot(n)$ is a linear function between any two consequent points with integer abscissas. From (2.10) and (1.3), we thus have that $S_\cdot(n)[\omega]$ and $\hat{S}_\cdot(n)[\omega]$ have the same shape and the former may be obtained from the latter by shifting by $\hat{S}_0(n)[\omega]$ in the vertical direction:

$$S_x(n)[\omega] = \hat{S}_x(n)[\omega] - \hat{S}_0(n)[\omega], \qquad \forall n \in \mathbb{N}, \quad \omega \in \Omega, \quad x \in \mathbb{R} \tag{2.11}$$

Obviously, each one of $S_\cdot(n)$ and $\hat{S}_\cdot(n)$ provides us with a complete picture of the distribution of particles and their velocities at epoch n in the BA. Theorem 2 will give the limit laws for these two functions. Before we formulate it, we pause to define the process Ψ which will be used to describe these laws.

The construction of Ψ makes use of a process G_t, $t \geqslant 0$, whose informal definition may be given as follows. $G_0 \equiv 0$, it has nondecreasing trajectories, and the Lebesque measure of the abscissas where the derivative is not zero is zero. Its generic trajectory is like the Cantor function and consists of plateaus. The lengths of the plateaus of $G_\cdot$ whose height (in $G_\cdot$) belongs to $[a, b]$ are independent of those whose height belongs to $[c, d]$, $0 \leqslant a < b < c < d < \infty$. If we consider those plateaus whose height is in

$[a, b]$, $0 \leqslant a < b < \infty$, then the number of those whose length exceeds l is a Poisson random variable with parameter $(b-a) \times [1/(\sqrt{\pi l}) - 1/\sqrt{\pi}]$, $l \in (0, 1]$.

There is a standard way to introduce $G_.$. We define a process T_s, $s \geqslant 0$, as the subordinator (Chapter 6 of ref. 7 contains all the facts about subordinators we will use below) whose Lévy measure μ has the following form:

$$\mu(dl) = \frac{dl}{2(\pi l^3)^{1/2}}, \qquad l \in (0, 1), \qquad dl \subset [0, 1] \tag{2.12}$$

We then define G_t, $t \geqslant 0$, as the process whose right-continuous inverse is $T_.$.

Let $\{G_t^{+i}, G_t^{-i}, t \geqslant 0\}_{i \in \mathbb{Z}}$ be a set of independent copies of the process G_t, $t \geqslant 0$. Let also $\hat{\theta}$ be the random variable such that

$$\mathbb{P}[\hat{\theta} \leqslant x] := 2x^{1/2}(1+x)^{-1}, \qquad x \in [0, 1] \tag{2.13}$$

and let $\{\hat{\theta}_i\}_{i \in \mathbb{Z}}$ be a set of independent copies of $\hat{\theta}$. Finally, let $\{\alpha_i, \beta_i\}_{i \in \mathbb{Z}}$ be a set of independent exponential mean-1 random variables.

By $(\Gamma, \mathscr{G})$ we denote the abstract probability space on which the random variables introduced above take their values in such a way that they all are mutually independent. On this space, we define a process $\Psi_.$ with continuous trajectories in the following way: To each $\gamma \in \Gamma$ associate the set $\{\zeta_i^+ = \zeta_i^+[\gamma], \zeta_i^- = \zeta_i^-[\gamma]\}_{i \in \mathbb{Z}}$ of hitting times:

$$\zeta_i^+[\gamma] := \min\{t : G^{+i}(t)[\gamma] = \alpha_i[\gamma]\}$$

$$\zeta_i^-[\gamma] := \min\{t : G^{-i}(t)[\gamma] = \beta_i[\gamma]\}, \qquad i \in \mathbb{Z}$$

and construct the set of random variables

$$\{t_i = t_i[\gamma], t_i' = t_i'[\gamma], t_i''[\gamma], t_i''' = t_i'''[\gamma]\}_{i \in \mathbb{Z}}$$

by $t_0 = 0$ and

$$\begin{aligned} t_i' &= t_i + \zeta_i^+ \\ t_i'' &= t_i' + \hat{\theta}_i \\ t_i''' &= t_i'' + \zeta_i^- \\ t_{i+1} &= t_i''' + 1, \qquad i \in \mathbb{Z} \end{aligned} \tag{2.14}$$

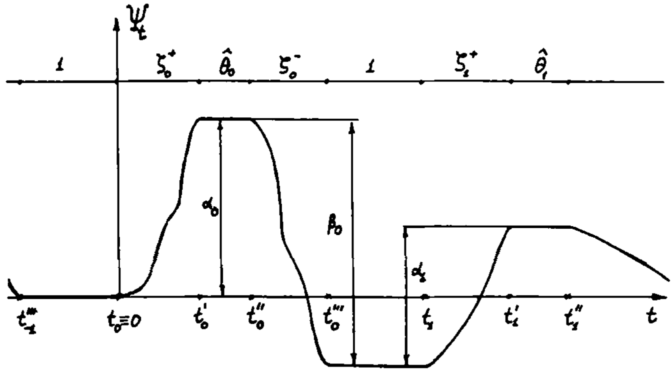

Fig. 1. A generic trajectory of the process $\Psi_{\cdot}$.

Then the realization of the process $\Psi_{\cdot}$ on an element $\gamma \in \Gamma$ has the following structure (see Fig. 1):

$$
\begin{aligned}
&\Psi_0 := 0 && \\
&\Psi_{t_i + t} - \Psi_{t_i} := G_t^{+i}, && 0 \leqslant t \leqslant \zeta_i^+ \\
&\Psi_{t_i' + t} - \Psi_{t_i'} := 0, && 0 \leqslant t \leqslant \hat{\theta}_i \\
&\Psi_{t_i'' + t} - \Psi_{t_i''} := -G_t^{-i}, && 0 \leqslant t \leqslant \zeta_i^- \\
&\Psi_{t_i''' + t} - \Psi_{t_i'''} := 0, && 0 \leqslant t \leqslant 1
\end{aligned}
\tag{2.15}
$$

We are now in a position to formulate our result. Observe, that $(M_{t/2}B)_{\cdot}$ is just an alternative notation for $B_{\cdot}^{\min, t}$ defined in (1.6). We also remark that $[nt]$, the integer part of nt, appears below in $n^{-1/2}\hat{S}_{2bn.}([nt])$ because formally we have defined the BA process solely for natural instances of time.

Theorem 2. (i) Let B_x, $x \in \mathbb{R}$, be a standard two-sided, one-dimensional Brownian motion with $B_0 = 0$, a.e. Then, for every $t > 0$, $(n)^{-1/2}\, \hat{S}_{2n.}([nt])$ converges, as $n \to \infty$, weakly on every finite interval of $\mathbb{R}$, to the process $\sqrt{2}\,(M_{t/2}B)_{\cdot}$ (or, equivalently, to the process $\sqrt{2}\, B_{\cdot}^{\min, t}$).

(ii) The process $(n)^{-1/2}\, S_{2n.}(n) = (n)^{-1/2}\, (\hat{S}_{2n.}(n) - \hat{S}_0(n))$ conditioned to the event $\{X_0(n) = 1 \text{ and } X_{L(0)}(n) = -1\}$ [$L(\cdot)$ has been defined right after Theorem 1] converges as $n \to \infty$, weakly on every finite interval of $\mathbb{R}$, to the process $\Psi_{\cdot}$.

We now devise a terminology which will be handy in describing the shape of trajectories of $S_{\cdot}(n)$ and $\hat{S}_{\cdot}(n)$. A portion of $S_{\cdot}(n)[\omega]$ whose abscissas belong to an interval $[x, y]$ will be called *nondecreasing* (*nonincreasing*) if $[x, y)$ contains only those particles of $X(n)[\omega]$ which have positive (negative) velocities, the sites x and $y-1$ are occupied by particles with positive (negative) velocity, and the nearest particle to the left of x as well as the nearest particle to the right of $y-1$ have negative (positive) velocities. A portion of $S_{\cdot}(n)$ whose abscissas belong to an interval $[z, u]$ is called a *valley* (*plateau*) if z is the abscissa of the rightmost point of a nonincreasing (nondecreasing) portion, u is the abscissa of the leftmost point of a nondecreasing (nonincreasing) portion, and there are no particles in (z, u). A portion of $\hat{S}_{\cdot}(n)[\omega]$ will have the same name as that portion of $S_{\cdot}(n)[\omega]$ with which it coincides after $\hat{S}_{\cdot}(n)[\omega]$ is shifted by $\hat{S}_0(n)[\omega]$ in the vertical direction [recall (2.11)]. The division of $(n)^{-1/2}\, S_{2n\cdot}(n)[\omega]$ into portions is naturally inherited from the division of $S_{\cdot}(n)[\omega]$.

Utilizing the above terminology, we may interpret the result (ii) of Theorem 2 as the description of the limit law for $S_{\cdot}(n)$ (under appropriate rescaling) as seen from a point at which a plateau and a nondecreasing portion of $S_{\cdot}(n)$ join.

Remark 1. It was conjectured in ref. 8 that the process to which $(n)^{-1/2}\, S_{2n\cdot}(n)$ converges should be self-similar with appropriate scaling factors. More precisely, the distribution of this process was conjectured not to change if all the trajectories are shrunk by $b^{1/2}$ along the ordinate and by b along the abscissa for any $b>0$. This conjecture indeed holds and it may be demonstrated by showing that $(nb)^{-1/2}\, S_{2nb\cdot}(n)$ conditioned to the event $\{X_0(n)=1,\, X_{L(0)}(n)=-1\}$ converges to $\Psi_{\cdot}$ as $n \to \infty$. This proof will be analogous to the one we present in this paper to establish (ii) of Theorem 2. However, the self-similarity of the limit law can be seen without its explicit calculation, but rather from the self-similarity of the Brownian motion. The argument is as follows: Fix $b \in \mathbb{N}$ to avoid technical complications. Observe that $\hat{S}_{2nb\cdot}(n)$ should be distributed as $\hat{S}_{2n\cdot}(nb)$ because "speeding up" the time by b in the BA is equivalent to changing the velocities from ± 1 to $\pm b$. Thus, due to (i) of Theorem 2, $(nb)^{-1/2}\, \hat{S}_{2bn\cdot}(n)$ converges to $(2/b)^{1/2}\, (M_{b/2}B)_{\cdot}$. Using then the fact that $b^{1/2}B_{\cdot}$ is distributed as $B_{b\cdot}$, it is easy to derive that $(2/b)^{1/2}\, (M_{b/2}B)_{\cdot} = \sqrt{2}(M_{1/2}B)_{\cdot}$ in distribution. The argument is then completed by recalling that $S_{\cdot}(n)$ and $\hat{S}_{\cdot}(n)$ have the same shape [relation (2.11)].

Remark 2. As we observed in Section 1, the proof of item (i) of Theorem 2 follows by a simple argument based on a specific relation of $S_{\cdot}(n)$ to $S_{\cdot}(0)$ and the fact that $(n)^{-1/2}\, S_{2n\cdot}(0)$ converges to $\sqrt{2}\, B_{\cdot}$. We did not find, however, the law of the process $(M_{1/2}B)_{\cdot}$ in the literature. The explicit description of the limit law for $S_{\cdot}(n)$ we gave in item (ii) of

Theorem 2 follows from the results of Theorem 1. One may then combine (i) and (ii) of Theorem 2 to conclude the following statement (whose proof will be omitted).

Statement. For a function $f: \mathbb{R} \to \mathbb{R}$, call a point $(x, f_x) \in \mathbb{R}^2$ a corner point if $f_z = f_x$ for all $z \in [x-1, x]$ and there exists $\varepsilon > 0$ such that $f_z > f_z$ for all $z \in (x, x+\varepsilon)$. Then $\sqrt{2}(M_{1/2}B)_.$ (equivalently, $\sqrt{2}\, B^{\min, l}_.$) conditioned to have a corner point at $(0, 0)$ is distributed as $\Psi_.$.

One may consider the above corollary as a result on a Brownian motion obtained via its approximation by a simple symmetric random walk on $\mathbb{Z}$. It is natural to expect that the same result could be derived without employing this random walk, but rather directly from the definition of the Wiener measure. We have no a complete and clear idea of how this could be done (see, however, Remark 3 below). What we conjecture is that the structure of the nondecreasing portions of $\Psi_.$ may be captured employing a reasoning similar to that presented in ref. 7 to obtain the law of $B^{\max}_.$, where for each $x \geqslant 0$, $B^{\max}_x := \max\{B_y, 0 \leqslant y \leqslant x\}$. Our conjecture is suggested by the following observation. It is easy to calculate from the results presented in ref. 7 that the Lévy measure for the right-continuous inverse of the process $\sqrt{2}\, B^{\max}_.$ is

$$\mu^{\max}(dl) = \frac{dl}{2(\pi l^3)^{1/2}}, \qquad l \in (0, +\infty)$$

We thus observe that $\mu^{\max}$ coincides with μ on $(0, 1)$. In words, the law of $G_.$ is that of $\sqrt{2}\, B^{\max}_.$ conditioned to have no flat portions of length greater than 1.

Remark 3. After this paper was composed, Jim Pitman pointed out that the distribution of $B^{\min, l}_.$ can be obtained using methods of Palm/excursion theory as developed in refs. 9 and 10 and some of his subsequent unpublished work.

3. PROOFS

3.1. Distribution for Finite Time

Remark on Notation. We will usually write X_i for $X_i(0)$.
Define Z_i, $i \in \mathbb{Z}$, by

$$Z_0 \equiv 0; \qquad Z_i = Z_{i-1} + X_i, \qquad i \in \mathbb{Z} \tag{3.1}$$

Observe that due to (1.2),

$$Z_i,\ i \in \mathbb{N}, \qquad \text{and} \qquad Z_{-i},\ i \in \mathbb{N} \tag{3.2}$$

are simple random walks in $\mathbb{Z}$ starting from 0.

We call a particle *positive* (*negative*) if its velocity is $+1$ (-1). Particles annihilate in pairs. Particles of the same pair are said to be *annihilating companions* of each other.

The following assertion will be frequently exploited in the sequel. It is a direct consequence of the dynamics of BA [rules (i), (ii) from Section 1].

Assertion 1. (i) Denote by i^* the initial position of the annihilating companion of the particle which initiated from the site i. Then, independently of i, these particles have annihilated by time n iff $|i-i^*| \leqslant 2n-1$. If these particles have annihilated by time n, then so have all the particles which were initially present between the sites i and i^*.

(ii) If a positive particle occupies initially a site i, then the initial position of its annihilation companion is the first return of Z_k, $k>i$, to the level $Z_{i-1} = Z_i - 1$.

Proof of Theorem 1. Throughout the proof, n is assumed to be arbitrarily fixed.

If for some $\omega_1, \omega_2 \in \Omega$, $X(0)[\omega_1]$ may be obtained from $X(0)[\omega_2]$ by a translation of $\mathbb{Z}$, then the same tranlation brings $X(n)[\omega_2]$ to $X(n)[\omega_1]$ for all $n \in \mathbb{N}$. Combining this with the fact that the law of $X(0)$ is translation invariant, one obtains (i) of the theorem.

Let us demonstrate (ii). Take $C = \{X_i(n) = 1\}$; for another choice of C, the proof is analogous. It is easy to derive from the dynamics of BA that when $A \cap C \neq \varnothing$ and $B \cap C \neq \varnothing$, then $B \cap C$ is uniquely expressed through the values of $X_k(0)$, $k \geqslant i-n$, while A is uniquely expressed through the values of $X_k(0)$, $k < i-n$. (To see that the assumptions $A \cap C \neq \varnothing$ and $B \cap C \neq \varnothing$ are indispensable, observe that if, for example, $A = \{X_{i-1}(n) = -1\}$, then there is no $X(0)[\omega]$ such that $X(n)[\omega] \in A \cap C$). Since $X_i(0)$, $i \in \mathbb{Z}$, are independent, then it holds that

$$\mathbb{P}[A \cap C \cap B] = \mathbb{P}[A]\ \mathbb{P}[C \cap B] \tag{3.3}$$

Taking $B = \Omega$ in (3.3) gives that $\mathbb{P}[A \cap C] = \mathbb{P}[A]\ \mathbb{P}[C]$. Combining the latter with (3.3), one easily derives both equalities of (2.5).

Now, we start the argument which will establish (iii) and (iv). Call the particle which occupies the site 1 at time 0 *principal*. Define $\Omega^+ = \Omega^+(n) := \{\omega \in \Omega : X_{n+1}(n)[\omega] = 1\}$ and $\Omega^- = \Omega^-(n) := \{\omega \in \Omega : X_{-n+1}(n)[\omega] = -1\}$. Ω^+ and Ω^- are the events that the principal particle has velocity $+1$ and -1, respectively, and is alive at time n. The particle which is the closest particle to the rigth of the principal particle at time n will be called *successor*. (Observe that the successor is defined on the set $\Omega^+ \cup \Omega^-$.) Denote by $\Omega^{\cdot\pm} = \Omega^{\cdot\pm}(n)$ the event that the successor has velocity ± 1. On the set $\bigcup^* \Omega^a \cap \Omega^{\cdot b}$, where the union $\bigcup^*$ extends over all ordered pairs (a, b), a, $b \in \{+, -\}$, we introduce the random variable $\Delta = \Delta(n)$ as the distance between the principal particle and its successor at time n. For $a, b \in \{+, -\}$, we then define the random variable $\sigma^{ab} = \sigma^{ab}(n)$ by

$$\mathbb{P}[\sigma^{ab} = l] := \mathbb{P}[\Delta = l \mid \Omega^a \cap \Omega^{\cdot b}], \qquad l = 1, 2, \dots \tag{3.4}$$

σ^{ab} expresses the distance between the principal particle and its successor at time n given the principal particle has survived by time n and has velocity a and its successor has velocity b.

Define the set of random variables

$$k_1 \equiv 2, r_1, k_2 \equiv r_1 + 1, r_2, \dots, k_i \equiv r_{i-1} + 1, r_i, \dots \tag{3.5}$$

by the following rules: for $i > 1$, k_i is the position of the leftmost particle in the interval $[r_{i-1} + 1, \infty)$ in $X(0)$ [for an initial distribution other than (1.2), $k_i - r_{i-1}$ may not be a degenerate random variable as it happens in the considered case]; for $i \geqslant 1$, r_i is the initial position of the annihilating companion of the particle which originated from k_i. Using the random variables (3.5), we define the set of events $\{A_m = A_m(n),\ m \geqslant 1\}$ by

$$A_m := \{X_{k_m} = -1 \text{ and } X_{k_i} = 1, r_i - k_i < 2n \text{ for } i = 1, \dots, m-1\} \tag{3.6}$$

Verbally, A_m consists of those $\omega \in \Omega$ for which initially there is a positive particle at the site 2 and it will have annihilated by time n, and also immediately to the right of its annihilation companion there is a positive particle which as well will have annihilated by time n, and this situation repeats recursively $m-1$ times, and, finally, there is a negative particle immediately to the right of the negative particle of the $(m-1)$th pair. It follows from Assertion 1 and our definition of A_m that on the event $\Omega^- \cap A_m$, the successor is the particle which originated from k_m, its velocity is -1, and, thus,

$$\Delta = k_m - 1 = 1 + \sum_{i=1}^{m-1} (1 + r_i - k_i) \qquad \text{on} \quad \Omega^- \cap A_m \tag{3.7}$$

Observe that due to (1.2), (3.1), (3.2), and Assertion 1, we have that

$$\mathbb{P}[\{X_{k_i} = 1\} \cap \{r_i - k_i = 2l - 1\}]$$
$$= \mathbb{P}[Z_0 = 0, Z_1 > 0, \dots, Z_{2l-1} > 0, Z_{2l} = 0] = \frac{f_{2l}}{2} \tag{3.8}$$

for every k_i and all $l=1, 2,...$. Then, using the fact that (see III.3 in ref. 4)

$$f_{2l}=u_{2l}(2l-1)^{-1}=u_{2l-2}-u_{2l}, \qquad l=1, 2,... \tag{3.9}$$

and the independence of the random variables X_i, $i\in\mathbb{Z}$, we find that

$$\begin{aligned}\mathbb{P}[A_m]&=\mathbb{P}[X_{k_m}=-1]\prod_{i=1}^{m-1}\mathbb{P}[\{X_{k_i}=1\}\cap\{r_i-k_i<2n\}]\\&=\frac{1}{2}\left(\frac{1-u_{2n}}{2}\right)^{m-1}\end{aligned} \tag{3.10}$$

On the other hand, we have that

$$\begin{gathered}\Omega^-\cap\bigcup_{m=1}^{\infty}A_m=\Omega^-\cap\Omega^{\cdot -}\\ A_i\cap A_j=\varnothing, \qquad i\neq j\\ \mathbb{P}[\Omega^-\cap A_m]=\mathbb{P}[\Omega^-]\,\mathbb{P}[A_m]\end{gathered} \tag{3.11}$$

where for the first relation, one reasons as for (3.7), the second follows from the definition, and the last relation is true since Ω^- depends on X_i, $i\leqslant 1$, while A_m depends on X_i, $i>1$. Combining now (3.10) and (3.11), we derive the following:

$$\mathbb{P}[\Omega^{\cdot -}\mid\Omega^-]=(\mathbb{P}[\Omega^-])^{-1}\sum_{m=1}^{\infty}\mathbb{P}[\Omega^-\cap A_m]=\sum_{m=1}^{\infty}\mathbb{P}[A_m]=\frac{1}{1+u_{2n}} \tag{3.12}$$

Recall $L(i)$ [resp., R(i)] denotes the position of the first particle to the left (right) of the site i. A set of positive (negative) particles of $X(n)$ in a region $[a,b]\in\mathbb{Z}$ is called *a positive* (*negative*, respectively) *cluster* if $X_a(n)=X_b(n)=+1$ (-1), $X_k(n)\in\{0,1\}$ $(\{0,-1\})$ for all $k\in[a,b]$, and $X_{L(a)}(n)=X_{R(b)}(n)=-1$ $(+1)$.

For a, $b\in\{+,-\}$, define $p_n^{ab}:=\mathbb{P}[X_{R(i)}(n)=b\mid X_i(n)=a]$. Due to (i) of Theorem 1, p_n^{ab} does not depend on i. It follows from (3.1), (3.2), and Assertion 1 that $\mathbb{P}[\exists k{:}X_j(n)=0\ \forall j\geqslant k]=0$. Thus, $p_n^{a+}+p_n^{a-}=1$. Based on (ii) of Theorem 1 and using the two facts established above, we conclude that the number of particles in a cluster with a given velocity a is distributed as the random variable N_n below

$$\mathbb{P}[N_n=k]=(p_n^{aa})^k\,(1-p_n^{aa}), \qquad k=1, 2,... \tag{3.13}$$

Thus, if we assume that $p_n^{--} \neq p_n^{++}$, then from (3.13), the proportion of negative particles in $X(n)$ would be different from that of the positive ones. Since the latter is impossible, we have that $p_n^{--} \neq p_n^{++}$ and both are given by (3.12), which proves (iii) of the theorem.

We now start to prove (iv). First, from (3.8), (3.9), and (2.2), one has that for $l = 1, 3, \ldots, 2n-1$,

$$\begin{aligned}
&\mathbb{P}[\{r_i - k_i = l\} \cap \{X_{k_i} = 1, 1 \leqslant r_i - k_i \leqslant 2n-1\}] \\
&\quad = \mathbb{P}[X_{k_i} = 1, r_i - k_i = l] \\
&\quad = f_{l+1}/2 = \mathbb{P}[g_i(n) = l+1]\, \mathbb{P}[X_{k_i} = 1, 1 \leqslant r_i - k_i \leqslant 2n-1] \qquad (3.14)
\end{aligned}$$

Since both sides of (3.14) are zero when $l > 2n-1$ or when l is even, then (3.14) is true for all l. Using the dependence of the random variables X_i, $i \in \mathbb{Z}$, it is easy to check that for any sequence l_i, $i = 1, \ldots, m-1$,

$$\begin{aligned}
&\mathbb{P}[\{r_i - k_i = l_i, i = 1, \ldots, m-1\} \cap A_m] \\
&\quad = \prod_{i=1}^{m-1} \mathbb{P}[\{r_i - k_i = l_i\} \cap \{X_{k_i} = 1, 1 \leqslant r_i - k_i \leqslant 2n-1\}]\, \mathbb{P}[X_{k_m} = +1] \\
&\qquad\qquad (3.15)
\end{aligned}$$

Using (3.8), (3.14), and (3.15) and the independence between X_i, $i \in \mathbb{Z}$, and consequent independence of $r_i - k_i$, $i = 1, \ldots, m-1$, among themselves and of Ω^-, we conclude that for all $j \in \mathbb{N}$ [below, $g_i = g_i(n)$ and $\lambda = \lambda(n)$ defined by (2.2) and (2.3)]

$$\mathbb{P}\left[1 + \sum_{i=1}^{m-1} (1 + r_i - k_i) = j \mid \Omega^- \cap A_m\right] = \mathbb{P}\left[1 + \sum_{i=1}^{m-1} g_i = j\right] \qquad (3.16)$$

Next, from (3.11), (3.12), and (3.10) we derive that for all $m \geqslant 1$,

$$\begin{aligned}
\mathbb{P}[\Omega^- \cap A_m \mid \Omega^- \cap \Omega'^-] &= \left(\sum_{i=1}^{\infty} \mathbb{P}[A_i]\right)^{-1} \mathbb{P}[A_m] \\
&= \frac{\mathbb{P}[A_m]}{p_n} = \mathbb{P}[\lambda = m-1] \qquad (3.17)
\end{aligned}$$

From (3.16) and (3.17), we finally have that for all $j \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}[\sigma^{--} = j] \\
&\quad = \mathbb{P}[\Delta = j \mid \Omega^- \cap \Omega'^-] = \sum_{m=1}^{\infty} \mathbb{P}[\Delta = j \mid \Omega^- \cap A_m] \\
&\quad = \mathbb{P}[\Omega^- \cap A_m \mid \Omega^- \cap \Omega'^-]
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \mathbb{P}\left[\left(1+\sum_{i=1}^{m-1} (1+r_i-k_i)\right)=j \mid \Omega^- \cap A_m\right]$$
$$\times \mathbb{P}[\Omega^- \cap A_m \mid \Omega^- \cap \Omega'^-]$$
$$= \mathbb{P}\left[1+\sum_{i=1}^{\lambda} g_i = j\right] = \mathbb{P}[\sigma(n)=j] \tag{3.18}$$

Indeed, the first equality follows from the definition, the second equality is due to (3.11), the third one is based on (3.7), the fourth is valid because of (3.16), (3.17), and because $\lambda(n)$ is independent of $\{g_i(n),\ i=1,2,...\}$ by the definition, and, finally, the last equality holds because of (2.1). Recalling the meaning of σ^{--} and applying the property (i) of Theorem 1, one gets the first line of (2.7) from (3.18).

We now show that $\sigma^{--}=\sigma^{++}$ in distribution, which will establish the second line of (2.7). For every $i, j \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\mathbb{P}[X_i(n)=X_{R(i)}(n)=-1,\ R(i)-i=k]$$
$$=\mathbb{P}[X_{L(j)}(n)=X_j(n)=-1, j-L(j)=k]$$

To see it, express both events in terms of $X(0)$ and observe that they can be obtained one from another by a translation of $\mathbb{Z}$. But the last probability equals $\mathbb{P}[X_j(n)=X_{R(j)}(n)=1,\ R(j)-j=k]$ because for any set of indices I and any sequence $\{x_i\}_{i\in I}$, $\mathbb{P}[X_i=x_i,\ i\in I]=\mathbb{P}[X_i=-x_i,\ i\in I]$ and because of (i) of Theorem 1. This reasoning leads to the desired relation $\sigma^{--}=\sigma^{++}$.

Next, we sketch the proof of the third line of (2.7). We introduce the set of events $\{B_m=B_m(n),\ m\geqslant 1\}$ by

$$B_m := \{X_{k_i}=1,\ r_i-k_i<2n \text{ for } i=1,..., m-1,\ X_{k_m}=1,\ r_m-k_m\geqslant 2n\} \tag{3.19}$$

which are defined by the means of the r.v. (3.5). By analogy with A_m, it is easy to see that on the set $\Omega^- \cap B_m$, the particle which originated from k_m is the successor (of the principal particle at time n), its velocity is $+1$, and, thus

$$\Delta = 2n+k_m-1 = 2n+1+\sum_{i=1}^{m-1} (1+r_i-k_i)$$

on $\Omega^- \cap B_m$. [Observe that in contrast to (3.7), the term $2n$ appeared in the above expression since the particles have diverged by this amount during time n.] From the last relation for Δ, using the random variables (3.5), one finds the distribution of σ^{-+} in the same way as was done for σ^{--} from (3.7).

The devices necessary to establish the last line of (2.7) will be developed in the proof of the following lemma. The lemma's assertion itself will not be used to prove Theorem 1, but it will be of help when proving Theorem 2. The point is that due to (3.20), the mean and the variance of $\sigma(n)$, which we will need to estimate in that proof, are asymptotically close to those of $\xi(n)$. This fact will facilitate the calculations since the expression for $\xi(n)$ is simpler than that for $\sigma(n)$.

Lemma 1. For each $n \in \mathbb{N}$, it holds that

$$\mathbb{P}[\sigma(n)=k]=(1-u_{2n}^2)\,\mathbb{P}[\xi(n)=k]+u_{2n}^2\,\mathbb{P}\left[2n+\theta(n)+\sum_{j=1}^{\lambda(n)} g_j(n)=k\right] \tag{3.20}$$

for all $k \in \mathbb{N}$, where the random variable $\xi(n)$ is defined by

$$\mathbb{P}[\xi(n)=l-1]=f_l/(1-u_{2n}), \qquad l=2, 4,..., 2n \tag{3.21}$$

and the rest of the variables involved in (3.20) have been specified in Section 2.1.

Proof. We adopt all the definitions and notations introduced up to now. We will show that the r.h.s. of (3.20) expresses $\mathbb{P}[\sigma^{++}(n)=k]$. Together with the second line of (2.7) of Theorem 1 this will establish the lemma's assertion.

From (3.1) and Assertion 1,

$$\Omega^+=\{Z_0=0, Z_1>0,..., Z_{2n}>0\}, \qquad \text{whence, using (3.2),} \quad \mathbb{P}[\Omega^+]=\frac{u_{2n}}{2} \tag{3.22}$$

On the set Ω^+, define the random variable k_0 in the following manner: for every $\{X_i, i \in \mathbb{Z}\} \in \Omega^+$, k_0 is such that

$$Z_{k_0-1}=1, \qquad Z_{k_0}=2, \qquad Z_i \geqslant 2, \forall i \in [k_0, 2n] \tag{3.23}$$

Observe that

$$k_0 \qquad \text{assumes one of the values of the set} \quad \{2, 4,..., 2n\} \tag{3.24}$$

since if (3.24) is untrue, then at least one of $Z_2,..., Z_{2n}$ equals zero, contradicting the fact that this sequence is from Ω^+. Call *marked* the particle which occupies initially the site k_0. It follows from the definition that the velocity of the marked particle is $+1$. On the set Ω^+, define the random variable r_0 as the initial position of the annihilating companion of the marked particle. On the set Ω^+ we then introduce the sequence of random variables $\{k_i, r_i, i=1, 2,...\}$ in the following way: $k_1 := r_0 + 1$; and the rest of the variables are the same as those from (3.5). Let $\{B_m = B_m(n), m = 1, 2,...\}$ be the events defined through these r.v. by (3.19).

Set $C_0 := \{r_0 - k_0 \geqslant 2n\}$. From (3.23), (3.24), and Assertion 1, it is easy to see that on the set $\Omega^+ \cap C_0$, the marked particle is the successor, its velocity is $+1$, and, thus, $\Delta = k_0 - 1$ on $\Omega^+ \cap C_0$. Consequently,

$$\begin{aligned}
&\mathbb{P}[\{\Delta = l-1\} \cap C_0 \cap \Omega^+] \\
&\quad = \mathbb{P}[\{k_0 = l\} \cap C_0 \cap \Omega^+] \\
&\quad = \mathbb{P}\{Z_0 = 0, Z_1 > 0,..., Z_{l-2} > 0, Z_{l-1} = 1, \\
&\qquad Z_l = 2, Z_{l+1} > 1,..., Z_{2n-1+l} > 1\} \\
&\quad = 2\mathbb{P}[Z_0 = 0, Z_1 > 0,..., Z_{l-1} > 0, Z_l = 0] \\
&\qquad \times \mathbb{P}[Z_0 = 0, Z_1 > 0,..., Z_{2n} > 0] \\
&\quad = 2\frac{1}{2} f_l \frac{1}{2} u_{2n} = \frac{f_l u_{2n}}{2}, \qquad l = 2, 4,..., 2n
\end{aligned} \tag{3.25}$$

Due to (3.24), we then have that

$$\mathbb{P}[\Omega^+ \cap C_0] = \sum_{l=2}^{2n} \mathbb{P}[\{k_0 = l\} \cap C_0 \cap \Omega^+] = \sum_{l=2}^{2n} f_l u_{2n}/2 = (1 - u_{2n})\, u_{2n}/2 \tag{3.26}$$

Also, by (3.22) and (iii) of Theorem 1,

$$\mathbb{P}[\Omega^+ \cap \Omega'^+] = \mathbb{P}[\Omega'^+ \mid \Omega^+]\, \mathbb{P}[\Omega^+] = p_n u_{2n}/2 \tag{3.27}$$

From (3.25)–(3.27) and the definition of $\xi = \xi(n)$ given in (3.21) we conclude that for $l = 2, 4,..., 2n$,

$$\mathbb{P}[\Delta = l-1 \mid \Omega^+ \cap C_0]\, \mathbb{P}[\Omega^+ \cap C_0 \mid \Omega^+ \cap \Omega'^+] = (1 - u_{2n}^2)\, \mathbb{P}[\xi = l-1] \tag{3.28}$$

Put now $\hat{C}_0 := \{r_0 - k_0 < 2n\}$. Assume that $\Omega^+ \cap \hat{C}_0 \cap B_m$ holds. Then, from the definition of the marked particle and Assertion 1 it follows that all the particles which were initially present in the region $[2, r_0]$ will have annihilated each other by time n. From the definition of B_m, then, the particle which originated from k_m is the successor, its velocity is $+1$, and, thus,

$$\Delta = k_m - 1 = r_0 + \sum_{i=1}^{m-1} (1 + r_i - k_i) \qquad \text{on} \quad \Omega^+ \cap \hat{C}_0 \cap B_m \tag{3.29}$$

Observe that from (3.24) and Assertion 1, $r_0 - 2n \in \{1, 3,..., 2n-1\}$ and for any $l \in \{1, 3,..., 2n-1\}$,

$$\begin{aligned}
&\mathbb{P}[\{r_0 = 2n + l\} \cap \Omega^+ \cap \hat{C}_0] \\
&\quad = \sum_{k = l+1,...,2n} \mathbb{P}\{Z_0 = 0, Z_1 > 0,..., Z_{k-2} > 0, \\
&\qquad Z_{k-1} = 1, Z_k = 2, Z_{k+1} > 1,..., Z_{2n+l-1} > 1, Z_{2n+l} = 1\} \\
&\quad = \sum_{k = l+1,...,2n} 2\mathbb{P}[Z_0 = 0, Z_1 > 0,..., Z_{k-1} > 0, Z_k = 0] \\
&\qquad \times \mathbb{P}[Z_0 = 0, Z_1 > 0,..., Z_{2n+l-k} > 0, Z_{2n+l-k+1} = 0] \\
&\quad = \tfrac{1}{2}(f_{l+1} f_{2n} + f_{l+3} f_{2n-2} + \cdots + f_{2n} f_{l+1}) \\
&\quad = \mathbb{P}[\theta = l]\, \mathbb{P}[\Omega^+ \cap \hat{C}_0]
\end{aligned} \tag{3.30}$$

where the last equality follows from (2.4) and the fact that

$$\mathbb{P}[\Omega^+ \cap \hat{C}_0] = \mathbb{P}[\Omega^+] - \mathbb{P}[\Omega^+ \cap C_0] = u_{2n}^2/2$$

which is established using (3.22) and (3.26).

Next, we observe that for any sequence l_i, $i \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}[\{r_0 = l_0, r_i - k_i = l_i, i = 1,..., m-1\} \cap \Omega^+ \cap \hat{C}_0 \cap B_m] \\
&\quad = \prod_{i=1}^{m-1} \mathbb{P}[\{r_i - k_i = l_i\} \cap \{X_{k_i} = 1, 1 \leqslant r_i - k_i < 2n\}] \\
&\qquad \times \mathbb{P}[\{r_0 = l_0\} \cap \Omega^+ \cap \hat{C}_0]\, \mathbb{P}[\{X_{k_m} = 1, r_m - k_m \geqslant 2n\}]
\end{aligned} \tag{3.31}$$

Then, based on (3.14), (3.30), and (3.31) and using independence of the r.v. X_i, $i \in \mathbb{Z}$, we derive that for each $l \in \mathbb{N}$ [below, θ, g_j, and λ stand for, respectively, $\theta(n)$, $g_j(n)$, and $\lambda(n)$]

$$\mathbb{P}\left[r_0 + \sum_{i=1}^{m-1} (1 + r_i - k_i) = l \mid \Omega^+ \cap \hat{C}_0 \cap B_m\right] = \mathbb{P}\left[2n + \theta + \sum_{j=1}^{m-1} g_j = l\right] \tag{3.32}$$

Since,

$$\mathbb{P}[\Omega^+ \cap \hat{C}_0 \cap B_m] = \mathbb{P}[\Omega^+ \cap \hat{C}_0]\, \mathbb{P}[A_m \cap \{r_m - k_m \geqslant 2n\}]$$
$$= \mathbb{P}[\Omega^+ \cap \hat{C}_0]\, \mathbb{P}[A_m]\, u_{2n}$$

then from (3.29) and (3.32),

$$\mathbb{P}[\Delta = l \mid \Omega^+ \cap \hat{C}_0 \cap B_m]\, \mathbb{P}[\Omega^+ \cap \hat{C}_0 \cap B_m \mid \Omega^+ \cap \Omega^{\cdot +}]$$
$$= \mathbb{P}\left[2n + \theta + \sum_{j=1}^{m-1} g_j = l\right] \mathbb{P}[\Omega^+ \cap \hat{C}_0]\, \mathbb{P}[A_m]\, u_{2n} / \mathbb{P}[\Omega^+ \cap \Omega^{\cdot +}]$$
$$= u_{2n}^2 \mathbb{P}\left[2n + \theta + \sum_{j=1}^{m-1} g_j = l\right] \mathbb{P}[\lambda = m - 1] \tag{3.33}$$

where in the last passage, we used (3.27), (3.32), and the last equality of (3.17).

Now, using the same reasoning as one that led to (3.11), one finds that $C_0 \cap \hat{C}_0 = \varnothing$, $\Omega^+ \cap \hat{C}_0 \cap B_i \cap B_j = \varnothing$ if $i \neq j$, and

$$\Omega^+ \cap \Omega^{\cdot +} = \bigcup_{m=1}^{\infty} (\Omega^+ \cap \hat{C}_0 \cap B_m) \cup (\Omega^+ \cap C_0)$$

Consequently,

$$\mathbb{P}[\sigma^{++} = l] = \mathbb{P}[\Delta = l \mid \Omega^+ \cap C_0]\, \mathbb{P}[\Omega^+ \cap C_0 \mid \Omega^+ \cap \Omega^{\cdot +}]$$
$$+ \sum_{m=1}^{\infty} \mathbb{P}[\Delta = l \mid \Omega^+ \cap \hat{C}_0 \cap B_m]$$
$$\times \mathbb{P}[\Omega^+ \cap \hat{C}_0 \cap B_m \mid \Omega^+ \cap \Omega^{\cdot +}] \tag{3.34}$$

From (3.34), (3.33), and (3.28), we finally derive the assertion (3.20) of the lemma. ∎

To complete the proof of Theorem 1, we use the random variables k_i, r_i, $i \in \mathbb{N}$, which we defined in the proof of Lemma 1, and observe that on the set $\Omega^+ \cap \hat{C}_0 \cap A_m$, the successor is the particle which originated from k_m, its velocity is $+1$, and, thus, $\Delta = k_m - 1 - 2n = r_0 - 2n + \sum_{i=1}^{m-1} (1 + r_i - k_i)$ on $\Omega^+ \cap \hat{C}_0 \cap A_m$, where in contrast to (3.29) we subtracted $2n$ since the principal particle and its successor have converged by this amount by time n. Using the concept of the marked particle introduced in the proof of Lemma 1, it is easy to check that $\Omega^+ \cap \Omega^{\cdot -} = \Omega^+ \cap \hat{C}_0 \cap (\bigcup_{m=1}^{\infty} A_m)$ and $\Omega^+ \cap \hat{C}_0 \cap A_i \cap A_j = \varnothing$ when $i \neq j$. Proceeding as in the proof of the lemma, we finally derive that $\sigma^{+-} = \theta + \sum_{i=1}^{\lambda} g_i$ in distribution. This establishes the fourth line of (2.7) in (iv) of Theorem 1 and thus completes the proof of this theorem. ∎

3.2. The Limit Law

In this section, we show Theorem 2. We start with an auxiliary lemma.

Lemma 2. Let $\{\sigma_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{N}}$ be a set of i.i.d. random variables such that

$$\sigma_{n,k} = \frac{\sigma^{++}(n)}{2n} \quad \text{in distribution} \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{N} \tag{3.35}$$

For $n \in \mathbb{N}$, define a right-continuous nondecreasing process $T_y(n)$, $y \geqslant 0$, in the following way: $T_0(n) := 0$ and for each $i \in \mathbb{Z}^+$,

$$T_{i/\sqrt{n}}(n) := \sigma_{n,1} + \cdots + \sigma_{n,i}$$

while $T_{.}(n)$ is constant on the interval $[(i-1)/\sqrt{n}, i/\sqrt{n})$.

Then, $T_{.}(n)$ converges as $n \to \infty$, weakly on any finite interval of $[0, +\infty)$, to the process $T_{.}$ defined in Section 2.2.

Proof. Let $\xi_{n,k}$, $\theta_{n,k}$, $k, n \in \mathbb{N}$, be independent random variables such that

$$\begin{aligned} \xi_{n,k} &= \frac{\xi(n)}{2n}, \\ \theta_{n,k} &= \frac{2n + \theta(n) + \sum_{i=1}^{\lambda(n)} g_i(n)}{2n} \quad \text{in distribution} \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{N} \end{aligned} \tag{3.36}$$

We recall that the application of Stirling's formula [(9.15) in II.10 of ref. 4] gives

$$\left| u_{2n} - \frac{1}{\sqrt{\pi n}} \right| \leqslant \frac{1}{n\sqrt{n}} \quad \text{for all sufficiently large } n \tag{3.37}$$

From (3.21) and (3.36), using (3.9) and (3.37) and evaluating $\sum \sqrt{n}$ by $\int \sqrt{x}\, dx$, we then have that

$$\begin{aligned} \mathbb{E}[\xi_{n,1}] &= \frac{1}{2n(1-u_{2n})} \sum_{l=2}^{2n} (l-1) f_l = \frac{1}{2n(1-u_{2n})} \sum_{l=2}^{2n} u_l \\ &= \frac{1}{\sqrt{\pi n}} \left[1 + O\left(\frac{1}{\sqrt{n}} \right) \right] \end{aligned} \tag{3.38}$$

Let $G_{n,k}(G'_{n,k})$ be the distribution function of the random variable $\xi_{n,k} - \mathbb{E}\xi_{n,k}(\xi_{n,k}$, respectively). Then using (3.38),

$$\int_{-\infty}^{y} x^2\, dG_{n,k}(x) = \int_{-\infty}^{y+\mathbb{E}\xi_{n,k}} x^2\, dG'_{n,k}(x) - 2\mathbb{E}[\xi_{n,k}]\, \mathbb{E}[\xi_{n,k} I_{\{\xi_{n,k} \leqslant y + \mathbb{E}\xi_{n,k}\}}]$$
$$+ \mathbb{P}[\xi_{n,k} \leqslant y + \mathbb{E}\xi_{n,k}](\mathbb{E}\xi_{n,k})^2$$
$$= \int_{-\infty}^{y+\mathbb{E}\xi_{n,k}} x^2\, dG'_{n,k}(x) + O\left(\frac{1}{n}\right) \tag{3.39}$$

For $0 \leqslant y \leqslant 1$ (below, the square brackets in the upper limit of a sum mean "the integer part")

$$\int_{-\infty}^{y+\mathbb{E}\xi_{n,k}} x^2\, dG'_{n,k}(x) = [4n^2(1-u_{2n})]^{-1} \sum_{l=0}^{[2n(y+\mathbb{E}\xi_{n,k})]} (l-1)^2 f_l$$
$$= \frac{y^{3/2}}{3\sqrt{\pi n}} \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right] \tag{3.40}$$

where the second equality is obtained in a similar way to (3.38), taking into account the fact that $\mathbb{E}\xi_{n,k} = O(n^{-1/2})$. From the definition of $\xi_{n,k}$, the left-hand side of (3.40) equals 0 when $y < 0$ and equals $\int_{-\infty}^{1} x^2\, dG'_{n,k}(x)$, when $y > 1$ for all n large enough. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{[u\sqrt{n}]} \int_{-\infty}^{y+\mathbb{E}\xi_{n,k}} x^2\, dG'_{n,k}(x) = u \times F(y) \tag{3.41}$$

where

$$F(y) := \begin{cases} 0 & \text{if } y < 0 \\ (3\sqrt{\pi})^{-1} y^{3/2} & \text{if } 0 \leqslant y \leqslant 1 \\ (3\sqrt{\pi})^{-1} & \text{if } y \geqslant 1 \end{cases} \tag{3.41a}$$

Let now $H_{n,k}$ denote the distribution function of $\theta_{n,k} - \mathbb{E}\theta_{n,k}$. Since, by the definition, $|\theta(n)| \leqslant 2n$, then $\mathbb{E}\theta(n) < Cn$ and $\operatorname{Var}\theta(n) < Cn^2$. Also, reasoning as in (3.38), one easily gets that $\mathbb{E}g_i \sim C\sqrt{n}$ and $\mathbb{E}g_i^2 \sim Cn^{3/2}$. From the last two asymptotic relations, the distribution of $\lambda(n)$, and the fact that $\lambda(n)$ and $g_i(n)$, $i \in \mathbb{N}$, are independent, we have that $\operatorname{Var}(\sum_{i=1}^{\lambda} g_i) \sim \mathbb{E}\lambda \mathbb{E}(g_1^2)$. Using these facts and mutual independence of the random variables $\theta(n)$, $\lambda(n)$, and $g_i(n)$, $i \in \mathbb{N}$, we get from (3.36) that

$$\mathbb{E}\theta_{n,k} = O(1), \qquad \int_{-\infty}^{\infty} x^2\, dH_{n,k}(x) < \frac{C[\operatorname{Var}\theta(n) + \mathbb{E}\lambda(n)\, \mathbb{E}(g_1^2(n))]}{(2n)^2} = O(1) \tag{3.42}$$

Finally, let $F_{n,k}$ be the distribution function of $\sigma_{n,k} - \mathbb{E}\sigma_{n,k}$. Using the relation between $\sigma_{n,k}$, $\xi_{n,k}$, and $\theta_{n,k}$ which stems from (3.20) and (3.35), (3.36), we then derive on the basis of (3.38), (3.39), and (3.41) that as $n \to \infty$,

$$\begin{gathered}\sum_{k=1}^{[u\sqrt{n}]} \mathbb{E}\sigma_{n,k} \to \frac{u}{\sqrt{\pi}} \\ \lim_{n\to\infty} \sum_{k=1}^{[u\sqrt{n}]} \int_{-\infty}^{y} x^2\, dF_{n,k}(x) = \lim_{n\to\infty} \sum_{k=1}^{[u\sqrt{n}]} \int_{-\infty}^{y+\mathbb{E}\xi_{n,k}} x^2\, dG'_{n,k}(x)\end{gathered} \tag{3.43}$$

By (3.41) and (3.43), there is a constant $C > 0$ such that

$$\operatorname{Var}\left(\sum_{k=1}^{[u\sqrt{n}]} \sigma_{n,k}\right) = \sum_{k=1}^{[u\sqrt{n}]} \int_{-\infty}^{+\infty} x^2\, dF_{n,k}(x) \leqslant C$$

Thus, applying Theorem 2 of Section 21 of ref. 6, we conclude that

$$\begin{aligned}\mathbb{E}[\exp\{itT_u(n)\}] &= \mathbb{E}\left[\exp\left\{it\left(\sum_{k=1}^{[u\sqrt{n}]} \sigma_{n,k}\right)\right\}\right] \\ &\to \exp\left\{u\left(\frac{it}{\sqrt{\pi}} + \int_{-\infty}^{+\infty} [\exp(ity) - 1 - ity]\, y^{-2}\, dF(y)\right)\right\}\end{aligned} \tag{3.44}$$

for each $t \in \mathbb{R}$ as $n \to \infty$, where $F(y)$ was defined in (3.41a).

By a standard argument, we then conclude that the finite-dimensional distributions of $T_{\cdot}(n)$ converge, as $n \to \infty$, to that of a process $T_{\cdot}$ which is not decreasing, has independent increments, and $\mathbb{E}[\exp\{itT_u\}]$ is given by the expression in the right hand side of (3.44). Thus, $T_{\cdot}$ is a subordinator. From comparison of its characteristic function to the moment generating function in the Lévy–Khintchine representation of subordinators (see Theorem 6.2.7 in ref. 7), we find that the Lévy measure of $T_{\cdot}$ is μ defined in (2.12)..

To complete the proof, it is left to check that for each finite u, $\{T_y(n), 0 \leqslant y \leqslant u\}_{n\in\mathbb{N}}$ forms a tight family.[1] For this, consider the compact family of nondecreasing right-continuous functions from $[0, u]$ to $[0, K]$ and show using the expression for the characteristic function of $T_u(n)$ that its measure should be $> 1 - \varepsilon$ when K is chosen sufficiently large, uniformly in $n > n(\varepsilon)$. ∎

Recall the rule of division of $n^{-1/2}S_{2n.}(n)$ into portions which we introduced immediately after Theorem 2. Consider $n^{-1/2}S_{2n.}(n)$ conditioned to the event $\{X_0(n) = 1,\ X_{L(0)}(n) = -1\}$ and take that one of its nondecreasing portions whose leftmost point is at the origin of $\mathbb{R}^2$. Extend this portion to be defined on the whole $\mathbb{R}$ in a way such that the obtained process is stationary and has independent increments. This process will be exactly $G_x(n)$, $x \in \mathbb{R}^+$, studied in the lemma below.

Lemma 3. Using the random variables (3.35), define for each $n \in \mathbb{N}$,

$$\varsigma_{n,0} := 0, \qquad \varsigma_{n,k} := \sigma_{n,1} + \cdots + \sigma_{n,k}, \qquad k = 1, 2,...$$

For each $n \in \mathbb{N}$, define then the process $G_x(n)$, $x \geqslant 0$, with nondecreasing continuous trjectories in the following manner: $G_{\varsigma_{n,k}}(n) := k/\sqrt{n}$, $k \in \mathbb{N}$, $G_.(n)$ is constant on each interval $[\varsigma_{n,k} + 1/2n, \varsigma_{n,k+1}]$, $k \in \mathbb{N}$, and $G_.(n)$ is obtained by a linear interpolation between each pair of points $(\varsigma_{n,k}, G_{\varsigma_{n,k}}(n))$ and $(\varsigma_{n,k} + 1/2n, G_{\varsigma_{n,k}+1/2n}(n))$, $k \in \mathbb{N}$.

Then, $G_.(n)$ converges as $n \to \infty$, weakly on any finite interval of $[0, \infty)$, to the process $G_.$ defined in Section 2.2.

Proof. Fix $n \in \mathbb{N}$. Define $T^*_.(n)$ to be the right-continuous inverse of the process $G_.(n)$. Let $\mathbb{R}_1$ and $\mathbb{R}_2$ be two copies of $\mathbb{R}$. Consider the first quadrant of the Euclidean space $\mathbb{R}_1 \times \mathbb{R}_2$. Mark the points $0, \varsigma_{n,1}, \varsigma_{n,2},...$ on $\mathbb{R}_1$ and the ponts $0, 1/\sqrt{n}, 2/\sqrt{n},...,$ on $\mathbb{R}_2$ and consider the corresponding trajectory of $G_.(n)$ (as a function from $\mathbb{R}_1$ to $\mathbb{R}_2$) and that of $T^*_.(n)$ (as a function from $\mathbb{R}_2$ to $\mathbb{R}_1$). By the definition, $\varsigma_{n,k}$, $k \in \mathbb{N}$, determine uniquely the values of $\sigma_{n,k}$, $k \in \mathbb{N}$. Thus the marked points correspond to some trajectory of the process $T_.(n)$ whose construction was presented in Lemma 2. We consider this trajectory as a function from $\mathbb{R}_2$ to $\mathbb{R}_1$ and compare it now to the trajectory of $T^*_.(n)$. By our construction, $T^*_y(n) = T_y(n)$, when $y = i/\sqrt{n}$, $i \in \mathbb{N}$, and on each interval $[i/\sqrt{n}, (i+1)/\sqrt{n})$, $i \in \mathbb{N}$, $T_.(n)$ is constant while $T^*_.(n)$ grows linearly with the tangent $(2\sqrt{n})^{-1}$. Using this "closeness" between $T_.(n)$ and $T^*_.(n)$, the fact that $T^*_.(n)$ is a right-continuous inverse of $G_.(n)$, and $T_.$ is the right-continuous inverse of $G_.$ and the assertion of Lemma 2, we conclude that the finite-dimensional distributions of $G_.(n)$ converge, as $n \to \infty$, to that of $G_.$. The tightness of the family $\{G_.(n)\}_{n \in \mathbb{N}}$ is verified in the same way done in Lemma 2 for the family $\{T_.(n)\}_{n \in \mathbb{N}}$. ∎

Lemma 4. For $\theta(n)$ defined in (2.4) and $\hat{\theta}$ defined in (2.13), it holds that $\theta(n)/2n \to \hat{\theta}$ in distribution, as $n \to \infty$.

Proof. For each $l = 1, 2,..., n$, we have that $\mathbb{P}[\theta(n) \geqslant 2l - 1]$ equals

$$u_{2n}^{-2} \sum_{\substack{1 \leqslant i,j \leqslant n \\ i+j \geqslant n+l}} f_{2i} f_{2j} = \iint_{\substack{1 \leqslant x,y \leqslant n \\ x+y \geqslant n+l}} [\pi(2x-1)(2y-1)\sqrt{xy}]^{-1}\, dx\, dy + o(n^{-1}) \tag{3.45}$$

where the error of approximation of the sum by the integral was evaluated using (3.9) and (3.37). The value of the integral in (3.45) is

$$[(nl^{-1})^{1/2}-(ln^{-1})^{1/2}][\pi(n+l)]^{-1}-(l^{-1/2}-n^{-1/2})(\pi n^{1/2})^{-1}+o(n^{-1})$$

Thus, using the fact that $u_{2n}^{-2}=\pi n[1+O(n^{-1})]$ which follows from (3.37), we conclude that

$$\mathbb{P}\left[\frac{\theta(n)}{2n}\geqslant\frac{2l-1}{2n}\right]\to 1-\frac{2\alpha^{1/2}}{1+\alpha} \qquad \text{if} \quad \frac{l}{n}\to\alpha \text{ for some } \alpha\in(0,1) \text{ as } n\to\infty \quad \blacksquare$$

Proof of Theorem 2. Using the property $M_\alpha\circ M_b=M_{a+b}$, $a, b\geqslant 0$, and simple algebra, one finds easily that for all $x\in\mathbb{R}$, all $\omega\in\Omega$, and all $t\in\mathbb{N}$,

$$\frac{\hat{S}_{2nx}(tn)[\omega]}{\sqrt{n}}=\left(M_{t/2}\left(\frac{\hat{S}_{2n.}(0)[\omega]}{\sqrt{n}}\right)\right)_x$$

Since $n^{-1/2}S_{2n.}(0)[\omega]$ converges to $\sqrt{2}\,B_.$ and since $M_{t/2}$ is a continuous transformation, the assertion (i) for $t\in\mathbb{N}$ follows (see Theorem 5.1 in ref. 1). The way this assertion is extended to all $t\in\mathbb{R}$ is standard and will be omitted.

We start to establish (ii). Consider $S_.(n)$ conditioned to the event $\{X_0(n)=1,\ X_{L(0)}(n)=-1\}$. As we saw in the proof of Theorem 1, the number of positive particles in the cluster that includes the particle at the site 0 is expressed by the random variable N_n defined in (3.13). Let Y_n be the position of the rightmost particle in this cluster. Set $H_n:=N_n/\sqrt{n}$ and $\tau_n:=Y_n/(2n)$. Recall from Theorem 1 that N_n is independent of the distances between the particles in the cluster and that H_n converges to an exponential mean-1 random variable, facts we will use below. Observe that by our construction, the portion of $n^{-1/2}S_{2n.}(n)$ on $[0,\tau_n]$ may be considered as a part of $G_.(n)$ stopped at the moment it reaches the level H_n. Since by Lemma 3, $G_.(n)$ converges to $G_.$ and since the portion of $\Psi_.$ on $[0,t'_0]$ is obtained from $G_.$ by "stopping" it at a random level which has an exponential mean-1 distribution, we thus derive that the nondecreasing portion of $n^{-1/2}S_{2n.}(n)$ whose leftmost point is at the origin converges to the portion of $\Psi_.$ whose abscissas lie between $t_0=0$ and t'_0. (There are certain problems in determining the meaning of the convergence, since the abscissas of the considered portion of $n^{-1/2}S_{2n.}(n)$ belong to $[0,\tau_n]$ while those of the portion of $\Psi_.$ belong to $[0,t']$ and $t'_0\neq\tau_n$ with probability 1. It is thus natural to establish the convergence in the Skorohod topology. This can be easily done using the fact that $\tau_n\to t'$ and the fact that the distances between particles in a cluster are independent of the number of particles it contains; both facts are provided by Theorem 1.)

We next consider the plateau of $n^{-1/2}S_{2n.}(n)$ that follows immediately the nondecreasing portion considered above; our objective is to show it converges (in Skorohod topology) to the portion of $\Psi_.$ whose abscissas lie between the points t'_0 and t''_0. We have that the height and the length of the former are, respectively, H_n and $\theta(n)/2n$, and they are independent due to Theorem 1. The height and the length of the latter are represented by an exponential mean-1 random variable and the random variable $\hat{\theta}$, which are independent. Thus, the height and the length of the former converge to those of the latter due to Lemma 4 and the convergence of H_n mentioned above. Employing then the independence between these random variables, we achieve our objective.

We then study the nonincreasing portion of $n^{-1/2}S_{2n.}(n)$ that follows immediately the plateau considered in the above paragraph. Its leftmost point coincides with the rightmost one of this plateau, but its structure is inependent of the preceding portions of $(n)^{-1/2}S_{2n.}(n)$ (due to Theorem 1). Reasoning as above for the nondecreasing portion, one can show it converges to the portion of $\Psi_.$ whose abscissas lie between t''_0 and t'''_0.

Following the nonincreasing portion just considered, there is a valley whose length is $[2n+\sigma(n)]/2n$ independent of the preceding portions. Reasoning as for the plateau in the paragraph before the previous one, we derive its convergence to the portion of $\Psi_.$ between the abscissas t'''_0 and t_1.

Continuing reasoning in this way and "gluing" the portions of the processes, we get the final result, though in the Skorohod topology. But this leads to convergence in the space of continuous functions since all $S_.(n)$ and the limit process are continuous.[1] ∎

ACKNOWLEDGMENTS

This research was supported by FAPESP and CNPq. The presentation of the paper benefited from consultations with H. Kesten, C. Landim, J. Pitman, and M. Perman. We thank an anonymous referee for his comments.

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